Fundamental Complexity of Optical Systems

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***Abstract*—It is often claimed that future systems will necessarily be all-optical, because electronic devices are not fast enough to keep up with the increase in fiber capacity. However, two objections are commonly raised: first, optical systems need many basic optical components, which are typically very expensive; and second, optical systems need many switch reconfigurations, which are typically very slow.**

In this paper, we examine whether these two costs can be fundamentally bounded. First, we develop the equivalence between coding theory and optical system design by introducing the concept of super switches. Then, we show how the minimal expected number of switch reconfigurations is almost equal to the state space entropy of the optical system. Finally, we point out the trade-off between the two types of costs.

I. Introduction

Network systems such as routers, network processors or buffers are commonly implemented today using electronics. Consequently, their scaling abilities are limited by Moore's law and memory bandwidth growth, which have historically grown slower than optical link capacities [1]. As incoming fibers become faster, it becomes harder and harder for them to cope with the incoming traffic.

Because of this growth discrepancy, it is often claimed that future systems will necessarily be all-optical. However, two objections are commonly raised. First, optical systems need many basic optical components, which are typically very expensive. And second, optical systems need many switch reconfigurations, which are typically very slow, especially when compared to electronic switch reconfiguration times and packet arrival speeds. In this paper, our goals will first be to model these two types of costs, then present fundamental lower bounds on these costs in general optical systems, and finally build optical constructions that reach these lower bounds.

The research in this area started when Shannon published his paper on the memory requirements of a telephone exchange [2]. Then, the discussion on the complexity of connecting networks was further extended in [3]–[5], which introduced links to information theory. Also related are works on time-slot-interchange complexity [3], minimum-complexity combined time-space switching [6]–[8] and minimum-complexity optical queues [9], [10]. Below, we will consider and explain many results of these papers under the angle of the two types of costs.

In this paper, we first consider the question of minimizing the expected number of switch reconfigurations. We find equivalences between optical constructions and coding theory. In fact, we find that the question of designing an optical system with as few switch reconfigurations as possible is very similar to the question of designing an optimal code.

For each state of the system, we define the switches that take part in the formation of that state as *active*, and the switches that are irrelevant to the formation of that state as *passive*. We further define the *theoretical complexity* C^* of the system as the minimum expected number of active switches, where the expectation is taken over the state space. This helps us provide close lower and upper bounds on the theoretical complexity of the system. In fact, if we denote the entropy of the state space by H, we prove that

$$H \leq C^* \leq H + 1.$$

We then discuss the question of minimizing the number of 2×2 switches in the system. We show that if the number of possible states is K and the time it takes to perform one state is T, the practical complexity C is lower bounded as follows:

$$C \geq \lceil \frac{\log K}{T} \rceil.$$

A construction with a practical complexity that grows like this lower bound is said to be *practically optimal*. We show that there is a certain tradeoff between designing a system that is practically optimal, and a system that is theoretically optimal. This tradeoff appears usually when the states have a highly non-uniform distribution.

It is important to note that this paper mostly presents a fundamental approach to the complexity of optical systems, without putting a stress on practical implementations. By defining complexity and explaining how lower bounds on complexity can be obtained, it is laying the ground for more practical papers. In [11], we use these fundamental results to present constructions of optical buffers and optically-buffered routers that are practically optimal, i.e., have a number of basic components that grows like the practical complexity lower bound defined in this paper.

This paper is organized as follows. First, Section II presents definitions related to systems and complexity of systems. Then, Section III presents links between coding theory and the complexity of systems. The theoretical complexity of a construction is defined to be the theoretical minimum on the expected number of switch reconfigurations, where the expectation is calculated with respect to the states space. We find lower and upper bounds on the practical complexity of constructions, and define theoretically optimal constructions. Finally, Section IV, the practical complexity is defined to be the theoretical minimum on the number of 2×2 switches in a construction, and the tradeoff between the practical complexity and the theoretical complexity is presented.

II. Definitions

We will refer to a *system* as an ideal network element that has input links, output links and inner states. The outputs of the system are uniquely determined as a function of the inputs and the inner states of the system during the entire time of operation. Packet size is fixed, time is slotted and it takes one time-slot to transmit a packet. If packets have variable sizes, they are segmented into fixed size blocks during arrival and reassembled at departure. Let's first review the definition of *external states* and *internal states*, as defined in [3].

Definition 1: A system has a set T of external states, where each external state is a distinguishable possible system output.

Definition 2: A system has a set S of internal states, where each internal state is a different setting of the system elements.

Consider the mapping $\sigma : S \rightarrow T$, linking each internal state to the resulting external state. By causality, to each external state corresponds some internal state, i.e., σ is surjective. Therefore, in order to reach T different outputs, at least an equal number of internal states is required: $|S| \geq |T|$.

The external states are not necessarily equiprobable. Using their probability distribution, we are now able to define the system entropy.

Definition 3: Assume that there exists a probability distribution P_T on the set of external states T, with $\sum_{t_i \in T} P_T(t_i) = 1$. The *system entropy* is given by the entropy of the external states:

$$H = -\sum_{t_i \in T} P_T(t_i) \log(P_T(t_i))$$

Example 1: Consider the case of an $N \times N$ switch, where all the permutations are equiprobable. There are $N!$ equiprobable external states. Therefore, the entropy of an $N \times N$ switch is given by:

$$H = -\sum_{i=1,..,N!} \frac{1}{N!} \log(\frac{1}{N!}) = \log(N!)$$

The basic element in our optical constructions is an *optical* 2×2 *switch*.

Definition 4: An *optical* 2×2 *switch* is a network element with 2 inputs, 2 outputs and a control input c, see Figure 1. If $c = 0$, the switch is in a "bar" state, and the inputs are passed forward to the output links. If $c = 1$, the switch is in a "cross" state, and the outputs are the inputs with interchanged positions.

Definition 5: The *state duration* T_S of a system is the time it takes to form a single external state.

Example 2: The state duration of an unbuffered $N \times N$ switch is $T_S = 1$.

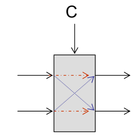

Fig. 1. A controlled 2×2 switch

III. Theoretical complexity and relations to coding theory

A. Complexity of optical systems

There is an intuitive connection between the construction of optical systems and coding theory [2], [3]. In fact, a 2×2 switch could be thought of as equivalent to a binary digit. In the same way as the binary digit can be set to zero or to one, the 2×2 switch can be set to a "cross" or a "bar" state. For instance, Shannon has famously argued that the number of 2×2 switches needed to construct an $N \times N$ switch able to realize all possible $N!$ permutations is at least $C = \log(N!)$ [2], which is exactly the number of digits required to code $N!$ symbols with uniform distribution.

We will use the connection between optimal coding and optimized optical construction to demonstrate an equivalence between the minimal expected code length and the minimal expected number of switch reconfigurations. While clearly not directly useful for practical implementations, this result is interesting in that it provides fundamental bounds on the complexity of optical systems.

B. Definition of theoretical complexity

In this section we will define a metric for the number of switch reconfigurations. We will denote it as the theoretical complexity, and present connections to coding theory.

We would first like to examine whether there is any connection between the entropy of the external states and the complexity of a network element, as would seem natural. The following example demonstrates that we should not measure the complexity of a network element only using its number of 2×2 switches, but also using some other metrics.

Example 3: Consider a system with N inputs: $(I_1, I_2, ..., I_N)$ and N outputs: $(O_1, O_2, ..., O_N)$, where N is even and $N \geq 4$. The system chooses between two equiprobable states: either the outputs are equal to the inputs, or the system interchanges the positions of every two consecutive inputs:

$$\begin{aligned}(O_1, O_2, ..., O_N) \in \quad & \{(I_1, I_2, I_3, I_4 ..., I_{N-1}, I_N) \\ & (I_2, I_1, I_4, I_3, ..., I_N, I_{N-1})\}\end{aligned}$$

In other words, the probability distribution over the set of all possible $N!$ permutations is given by:

$$P_T = \{\frac{1}{2}, 0, ..., 0, \frac{1}{2}, 0, ..., 0\}$$

The entropy of a system with two equiprobable states is $H = 1$. However, it is not possible to construct such a system with

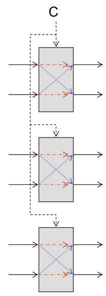

Fig. 2. A super switch

less than $\frac{N}{2}$ 2×2 switches, because there are N changeable inputs. Therefore, this simple example shows that the number of 2×2 switches of a system is not necessarily equal, or even close, to its entropy.

Let's introduce a new type of construction to bridge this apparent gap between entropy and complexity. As illustrated in Figure 2, a construction emulating the previous example is a column of $\frac{N}{2}$ switches, which are all controlled by the same control input. If the common control input is set to the "bar" state, the inputs are passed forward to the output. If the control input is set to the "cross" state, the locations of each pair of inputs are interchanged. We will define such a set of switches, which are all controlled by a single control input, as a *super switch*.

Definition 6: A *super switch* is an ensemble of 2×2 switches, which are all controlled by the same control input (see Figure 2). The number of 2×2 switches composing a super switch is called the *size* of the super switch.

The number of super switches in a construction equals the number of independent controls. These independent controls determine the internal states of the construction. Our goal will be to state a connection between the number of independent controls and the entropy of the network element.

In coding theory, the goal is to design a code such that the average length of codewords is minimized [12]. An optimal code is designed using the knowledge of the probabilities of source symbols.

It appears that the key in the connection between coding and switching lies in the number of independent controls introduced above: each digit in the codeword corresponds to one independent control. Similarly, each codeword corresponds to a given set of independent controls. However, the connection is not entirely straightforward: the number of digits in the codewords is not fixed, while the hardware in a construction is fixed. Therefore, we will define *active* and *passive* controls. For a specific external state, active controls are those that participate in forming that state, while the value of passive controls is irrelevant in forming the state.

Definition 7: A control input is called *active* for a specific external state if its value depends on the state. A control input is called *passive* if its value is predetermined independently of the state.

For instance, in the example above with the super switch (Figure 2), there is a single control input that is always active. If there were additional control inputs for switches that are never used, then these control inputs would be called passive.

Now we will define the *theoretical complexity* of an optical network element as the minimum expected number of active controls over all possible constructions emulating it. Our motivation will be to build optical constructions that reach that minimum.

Definition 8: Assume that there exists a probability distribution P_T on the set of external states T, and that l_{t_i} is the number of active controls necessary to form a state $t_i \in T$ in a given construction. The *theoretical complexity* C^* of a network element is the minimal expected number of active controls, where the minimum is taken over all possible emulating constructions:

$$C^* = \min \sum_{t_i \in T} P_T(t_i) l_{t_i}$$

For instance, consider the example above. There is a single control that is always active, therefore the expected number of active controls in the above construction is 1. This is an upper bound on the minimum expected number, thus $C^* \leq 1$. (In fact, we will see below that $C^* = 1$.)

We will now find lower and upper bounds on the theoretical complexity of a network element.

C. A lower bound on the theoretical complexity

The derivation of the lower bound on the theoretical complexity will parallel the derivation of the the lower bound on the expected length of codewords in coding theory [12].

A known result related to coding is Kraft's inequality. It states that if the length of the codewords for code L are $l_1, l_2, ..., l_m$, then it holds that $\sum_i 2^{-l_i} \leq 1$. Now, we can state an equivalent theorem also for switches. We will use this theorem in finding a lower bound on the theoretical complexity. (For the sake of presentation, full proofs of all the results in the paper are presented in [13].)

Lemma 1: (*Kraft's inequality*) Assume that the number of active controls for an external state $t_i \in T$ is given by l_{t_i}. It holds that

$$\sum_i 2^{-l_{t_i}} \leq 1.$$

By minimizing the expected number of active controls under Kraft's inequality, we get a lower bound on the theoretical complexity:

Theorem 1: Assume that the number of active controls for an external state $t_i \in T$ is given by l_{t_i} and that its probability is given by p_{t_i}. Then the theoretical complexity of the network element is lower bounded by its entropy:

$$C^* \geq H.$$

The idea in coding is to assign short codewords to the most frequent source words, and long codewords to the less frequent source words. In fact, it is possible to design a code such that the expected length of codewords is almost identical to the entropy of the source. Here, the idea is similar: states with high probability will be achieved with a small number of active switches, and states with lower probability will be achieved with a higher number of active switches.

D. An upper bound on the theoretical complexity

In this section, we present an upper bound on the theoretical complexity of memoryless network elements. The upper bound will be derived by showing that it is possible to construct a general optical system, where the active controls at each state are exactly the Huffman coding of that state. From coding theory, it is known that the expected length of codewords L of a code constructed by the Huffman procedure is upper bounded by $L \leq H+1$. Therefore, the expected number of active controls is similarly upper bounded. (A full and detailed proof can be found in [13].)

Theorem 2: The theoretical complexity is upper bounded as follows:

$$C^* \leq H+1$$

Now that we have close lower and upper bounds, we get a good measure of the optimality of a construction by using the expected number of active controls.

E. Theoretically optimal constructions

In most cases, it is extremely hard, if not impossible, to find a construction that fits the tight bounds on the theoretical complexity. Therefore, we will loosen the strict theoretical condition by only requiring that the expected number of active controls would grow like the theoretical complexity. Constructions satisfying this condition will be called *theoretically optimal*.

Definition 9: A construction is called *theoretically optimal* if its expected number of active controls is equal in growth to the theoretical complexity: $L = \Theta(C^*)$.

Let's show an example of a theoretically optimal construction emulating a switch with non equiprobable permutations.

Example 4: Consider an $N \times N$ switch, where two permutations have a very high probability and the rest of the permutations have a very low probability, i.e., the probabilities of the permutations are given by:

$$\pi = (\frac{1-\epsilon}{2}, \frac{1-\epsilon}{2}, \frac{\epsilon}{N!-2}, ..., \frac{\epsilon}{N!-2}),$$

where $0 < \epsilon \ll 1$. The entropy of this system is approximately given by:

$$H \approx 1 + \epsilon \log(N!)$$

Now, consider the construction in Figure 3. For simplicity of drawing, each switch represents a super switch of length N. This construction has three main stages. The first stage chooses between performing the first dominant permutation or continuing to the other options. The second stage chooses between performing the second dominant permutation or continuing to the other options. The third stage is a full Benes network [14], [15] that performs the permutations with the lower probability. Note that although the number of switches in this construction is almost similar to the number of switches in a regular Benes network, the number of switch reconfigurations is much lower in this construction. This is due to the fact that a Benes network is optimized to the case where all permutations are equiprobable, while here, the state probabilities are far from being equal. Therefore, the proposed construction using the a-priori knowledge of the state probabilities is more suitable.

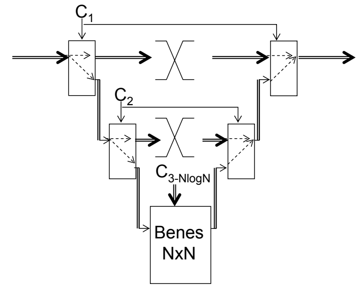

Fig. 3. A construction performing non-equiprobable permutations

We will now show that this construction is theoretically optimal. In order to do so, we will calculate the expected number of active controls with this construction:

$$\begin{aligned} L &= \sum_i \pi_i l_{t_i} \\ &= \frac{1-\epsilon}{2} * 1 + \frac{1-\epsilon}{2} * 2 + \epsilon * (N \log N + 2) \\ &\approx H + \frac{1}{2} \end{aligned}$$

Therefore this construction is theoretically optimal.

We have shown bounds on the theoretical complexity, defined theoretical optimality and shown examples for memoryless systems. In [13], we extend these results and prove that they also hold for the case of systems with memory.

Until now, the measure of optimality we were interested in was the expected number of active controls, which is related to the number of switch reallocations. Now, we will consider the question of minimizing the number of 2×2 switches in the construction.

IV. Practical Complexity versus Theoretical Complexity

A. Definition of practical complexity

Until now, we considered only the number of different states and their probabilities in calculating the theoretical complexity. The structure of states themselves had no significance. In this section, we will define the practical complexity of a network element as the number of 2×2 switches required to emulate a system, and illustrate the tradeoff between practical and theoretical complexity.

Definition 10: The *practical complexity* C of a construction is the number of 2×2 switches in the construction.

In order to find the connection between the number of states and the practical complexity, we will consider the state duration. We will assume that all the states have equal state duration. In order to find the connection between the number of states and the practical complexity we will use arguments similar to those mentioned in [5]. Consider the operation of C switches during T time-slots. If each switch is a single switch, i.e., not a part of a super switch, and is always active, the maximal number of internal states formed is 2^{CT}. Since the number of internal states is lower bounded by the number of external states, we get the following theorem:

Theorem 3: Assume that the number of different states in the construction is K. It holds that number of 2×2 switches C is lower bounded as follows:

$$C \geq \lceil \frac{\log K}{T} \rceil,$$

where T is the state duration.

Note that if the states are equiprobable, the theoretical complexity is $C^* = \log K$, and we get $C \geq \frac{C^*}{T}$. Now we can define a practically optimal construction to be a construction with a number of 2×2 switches that grows like the introduced lower bound.

Definition 11: Denote by C the practical complexity of a system with K states, and with state duration T. We say that a construction is *practically optimal* if $C = \Theta(\frac{\log K}{T})$, i.e., there is some constant a such that

$$\lceil \frac{\log K}{T} \rceil \leq C \leq a \frac{\log K}{T}$$

To illustrate this result, in [11], we present constructions of optical buffers and optically-buffered routers that are practically optimal. We first use the definitions above to provide a lower bound on the number of 2×2 switches required, and then present practical constructions with a number of 2×2 switches that grows like the introduced lower bound.

Definition 12: A construction that is both practically optimal and theoretically optimal will be called *optimal*.

Note that not every practically optimal construction is also a theoretically optimal construction, and not every theoretically optimal construction is also practically optimal. In fact, there often exists a tradeoff between theoretical and practical optimality. Let's present an example demonstrating the tradeoff.

Example 5: Consider the case of an $N \times N$ switch with two dominant permutations as presented in Example 4. On the one hand, emulating this system with a Benes network is practically optimal, but not necessarily theoretically optimal (because if the two dominant permutations are those from Example 3, it can easily be shown that a Benes network needs at least $N/2$ super switches). On the other hand, the construction presented in Figure 3, while theoretically optimal, would require more 2×2 switches than the Benes network.

The previous example illustrates the tradeoff between theoretical and practical complexity. Assuming that the reallocation of connections is an operation that requires time, we do not want to "waste" many reallocations on states that are very frequent. So for this example, a Benes network is highly wasteful and not theoretically optimal, even though it is practically optimal. On the other hand, note that a theoretically optimal construction for this case will require slightly more switches – although it is both practically optimal and theoretically optimal, and therefore optimal.

V. CONCLUSION

In this paper we were interested in two different cost measures: number of switches and expected number of switch reconfigurations. First, we presented links between switching theory and coding theory. We found that the design of a network element with a minimized expected number of switch reconfigurations is equivalent to the construction of optimal codes. Then, we presented a general construction of a switch that achieves cost lower bounds given some state probability distribution. Finally, we argued that such a construction is not always practical, and discussed the question of minimizing the number of 2×2 switches in the system.

As noted above, this paper lays a fundamental ground, by providing lower bounds on the complexity of practical implementations. In [11], we use these fundamental results to present constructions of optical buffers and optically-buffered routers that are practically optimal, i.e., have a number of basic components that grows like the practical complexity lower bound defined in this paper.

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